Controlling Complex Networks: How Much Energy Is Needed?

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The outstanding problem of controlling complex networks is relevant to many areas of science and engineering, and has the potential to generate technological breakthroughs as well. We address the physically important issue of the energy required for achieving control by deriving and validating scaling laws for the lower and upper energy bounds. These bounds represent a reasonable estimate of the energy cost associated with control, and provide a step forward from the current research on controllability toward ultimate control of complex networked dynamical systems.

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Complex networks are ubiquitous in natural, social, and man-made systems, such as gene regulatory networks, social networks, mobile sensor networks, and so on [1]. A network is composed of nodes and edges. The nodes represent individual units (e.g., genes, persons, sensors) and the edges represent connections or interactions between the nodes. The state of a node (e.g., protein being expressed, opinion of a person, position of a sensor) normally evolves over time. And the evolution depends not only on the node's intrinsic dynamics but also on the couplings with its nearest neighbors [2].

On one hand, the couplings between nodes increase the complexity of collective behaviors, which stimulates much interest of modeling, analyzing, and predicting dynamical processes on complex networks [3]. On the other hand, one may utilize the couplings to control a whole network, i.e., steering a network from any initial state (vector) to a desired final state, by driving only a few suitable nodes with external signals. In this direction there are good attempts recently from physics [4–10], biology [11,12], and engineering [13–16] research communities. Among others, Liu *et al.* studied the controllability of various real-world networks, i.e., the ability to steer a complex network as measured by the minimum number of driver nodes. A main result was that the number of driver nodes required for full control is determined by the network's degree distribution [8]. Issues such as achieving control by using only one controller [9,16] and making structural perturbations to the network to minimize the number of control inputs [10] have also been addressed.

When controlling a complex network, an important and unavoidable issue is the cost of control. For instance, in order to control a social network some effort has to be devoted to change a few individuals' opinions, while to control an electronic or a mechanical network, some energy has to be consumed to drive a few elements. Even if a network is controllable in principle, it may not be controllable in practice if it costs an infinite amount of energy or if it requires too much time to achieve the control. In this Letter, we address this outstanding issue of energy cost, i.e., the amount of effort or energy that is necessary to produce external signals for steering a complex network, and focus on its lower and upper bounds. Suppose a complex network is deemed to be controlled to a desired state in finite time T_f, our main results [see Eqs. (7) and (8)] show the scaling laws of the energy cost bounds with the control time T_f in two different regimes separated by the characteristic time. The results give faithful estimates for the required energy and thus can provide significant insights into bridging network controllability with actual control.

To be able to analyze the energy cost, we study linear networked systems subject to control inputs. This is the currently standard framework, upon which the network controllability analysis is built [6,8–10,16]. A typical system of N nodes and M controllers can be written as

$$\dot{\mathbf{x}}_t = \mathbf{A}\mathbf{x}_t + \mathbf{B}\mathbf{u}_t, \tag{1}$$

where $\mathbf{x}_t = [x_1(t), x_2(t), \ldots, x_N(t)]^T$ is the state vector of nodes, $\mathbf{u}_t = [u_1(t), u_2(t), \ldots, u_M(t)]^T$ is the input vector of external signals, $\mathbf{B} = \{b_{im}\}$ is the $N \times M$ input matrix with $b_{im} = 1$ if controller m connects to node i and $b_{im} = 0$ otherwise, $\mathbf{A} = \{a_{ij}\}$ is the weighted network's adjacency matrix including linear nodal dynamics $\{a_{ii}\}$.

The typical situation of controlling a complex dynamical network can be characterized as using external signals $\mathbf{u}_t$ to direct the system Eq. (1) from an arbitrary initial state $\mathbf{x}_0$ toward an arbitrary desired state $\mathbf{x}_{T_f}$ in the time interval $t \in [0, T_f]$. Assuming that the networked system is

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controllable [8,17], our goal is to obtain an analytic estimate of the energy cost required for achieving control, which is defined as [18] $\mathcal{E}(T_f) \equiv \int_0^{T_f} \|\mathbf{u}_t\|^2 dt$. Generally, an infinite number of possibilities exist for choosing the control input $\mathbf{u}_t$ to steer the system Eq. (1) from $\mathbf{x}_0$ to $\mathbf{x}_{T_f}$. Of all the possible inputs, the optimal control input is given by $\mathbf{u}_t = \mathbf{B}^T e^{\mathbf{A}^T(T_f - t)} \mathbf{W}_{T_f}^{-1} \mathbf{v}_{T_f}$, which minimizes the energy cost [18,19]. The corresponding minimized energy cost is then $\mathcal{E}(T_f) = \mathbf{v}_{T_f}^T \mathbf{W}_{T_f}^{-1} \mathbf{v}_{T_f}$, where $\mathbf{W}_{T_f} \equiv \int_0^{T_f} e^{\mathbf{A}t} \mathbf{B}\mathbf{B}^T e^{\mathbf{A}^T t} dt$ and $\mathbf{v}_{T_f} \equiv \mathbf{x}_{T_f} - e^{\mathbf{A}T_f} \mathbf{x}_0$ denotes the difference vector between the desired state under control and the final state during free evolution. For convenience, we set the origin as the desired state $\mathbf{x}_{T_f} = \mathbf{0}$ and rewrite the energy cost as

$$\mathcal{E}(T_f) = \mathbf{x}_0^T \mathbf{H}^{-1} \mathbf{x}_0, \tag{2}$$

where $\mathbf{H}(T_f) \equiv e^{-\mathbf{A}T_f} \mathbf{W}_{T_f} e^{-\mathbf{A}^T T_f}$ is the symmetric Gramian matrix [18]. When the system is controllable, $\mathbf{H}$ is positive definite (PD), otherwise it is noninvertible. In the following we focus on the normalized energy cost

$$E(T_f) = \mathcal{E}(T_f)/\|\mathbf{x}_0\|^2 = \frac{\mathbf{x}_0^T \mathbf{H}^{-1} \mathbf{x}_0}{\mathbf{x}_0^T \mathbf{x}_0}. \tag{3}$$

When $\mathbf{x}_0$ is parallel to the direction of one of $\mathbf{H}$'s eigenvectors, the corresponding inverse of the eigenvalue has the physical meaning of normalized energy cost associated with controlling the system along the particular eigendirection. Using the Rayleigh-Ritz theorem [20], we can bound the normalized energy cost as

$$\frac{1}{\eta_{\max}} \equiv E_{\min} \le E(T_f) \le E_{\max} \equiv \frac{1}{\eta_{\min}}, \tag{4}$$

where $\eta_{\max}$ and $\eta_{\min}$ are the maximal and minimal eigenvalues of the PD matrix $\mathbf{H}$, respectively.

To proceed, we focus on the lower and upper bounds of normalized energy cost for the case of single-node control. To analytically calculate the quantities $1/\eta_{\max}$ and $1/\eta_{\min}$, for weighted undirected networks, we decompose the matrix $\mathbf{A}$ in terms of its eigenvectors as $\mathbf{A} = \mathbf{V}\mathbf{S}\mathbf{V}^T$, where $\mathbf{V}$ is the orthonormal eigenvector matrix that satisfies $\mathbf{V}\mathbf{V}^T = \mathbf{V}^T\mathbf{V} = \mathbf{I}$, $\mathbf{S} = \mathrm{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_N\}$ with descending order $\lambda_1 > \lambda_2 > \ldots > \lambda_N$. We thus have $e^{\mathbf{A}t} = e^{\mathbf{A}^T t} = \mathbf{V} e^{\mathbf{S}t} \mathbf{V}^T$. Substituting these expressions into the Gramian matrix and noting that $\mathbf{V}$ is time independent, we have

$$\mathbf{H} = \mathbf{V} e^{-\mathbf{S}T_f} \left(\int_0^{T_f} e^{\mathbf{S}t} \mathbf{V}^T \mathbf{B}\mathbf{B}^T \mathbf{V} e^{\mathbf{S}t} dt \right) e^{-\mathbf{S}T_f} \mathbf{V}^T. \tag{5}$$

Denoting the only node under direct control as c, we have that $\mathbf{B}$ is an $N \times 1$ matrix, of which all elements are zeros except the cth element, which is one. After some amount of algebra, we obtain

$$H_{ij} = \sum_{\alpha=1}^{N} \sum_{\beta=1}^{N} \frac{V_{i\alpha} V_{c\alpha} V_{c\beta} V_{j\beta}}{\lambda_\alpha + \lambda_\beta} (1 - e^{-(\lambda_\alpha + \lambda_\beta)T_f}), \tag{6}$$

where the Roman letters i, j, c are node indices in the real space while the Greek letters α, β are running indices in the eigenspace.

To carry the analysis further, we note that there are two distinct regimes in terms of the control time T_f. In the small T_f regime where $T_f \ll 1/|\lambda_\alpha + \lambda_\beta|$, we can expand $e^{-(\lambda_\alpha + \lambda_\beta)T_f} \approx 1 - (\lambda_\alpha + \lambda_\beta)T_f$ and obtain $H_{ij} \approx T_f \sum_{\alpha=1}^{N} \sum_{\beta=1}^{N} V_{i\alpha} V_{c\alpha} V_{c\beta} V_{j\beta} = T_c \delta_{ic} \delta_{cj}$. In this case, we have $H_{ij} \approx 0$ for all i and j except $H_{cc} \approx T_f$ so that the maximal eigenvalue of matrix $\mathbf{H}$ can be approximated as T_f. Consequently, for the small T_c regime, we have $E_{\min} \equiv 1/\eta_{\max} \approx 1/T_f$, regardless of the form of the matrix $\mathbf{A}$ and of the value of c. In contrast, in the large T_f regime characterized by $T_f \gg 1/|\lambda_\alpha + \lambda_\beta|$, we can approximate the maximal eigenvalue of $\mathbf{H}$ by its trace, which has been numerically verified: $\eta_{\max} \approx \sum_{\alpha=1}^{N} \eta_\alpha \equiv \mathrm{Tr}[\mathbf{H}] = \sum_i^N \sum_\alpha^N \sum_\beta^N \frac{V_{i\alpha} V_{c\alpha} V_{c\beta} V_{i\beta}}{\lambda_\alpha + \lambda_\beta} (1 - e^{-(\lambda_\alpha + \lambda_\beta)T_f}) = \sum_{\alpha=1}^{N} \frac{V_{c\alpha}^2}{2\lambda_\alpha} (1 - e^{-2\lambda_\alpha T_f})$. If $\mathbf{A}$ is PD, the term $e^{-2\lambda_\alpha T_f}$ vanishes for large T_f. We thus have $E_{\min} \equiv 1/\eta_{\max} \approx 1/\sum_{\alpha=1}^{N} \frac{V_{c\alpha}^2}{2\lambda_\alpha} (1 - e^{-2\lambda_\alpha T_f}) \approx 1/\sum_{\alpha=1}^{N} \frac{V_{c\alpha}^2}{2\lambda_\alpha} = 1/[(\mathbf{A} + \mathbf{A}^T)^{-1}]_{cc}$. Note that, since the matrix $\mathbf{A}$ is independent of T_f, the factor $1/[(\mathbf{A} + \mathbf{A}^T)^{-1}]_{cc}$ is time independent too. This means that, when $\mathbf{A}$ is PD, the lower bound of the energy cost converges to a constant value for large T_f. If $\mathbf{A}$ is not PD, i.e., at least one of $\mathbf{A}$'s eigenvalues is negative, the most negative eigenvalue λ_N will dominate the behavior of $\mathbf{H}$: $H_{ij} \approx \frac{V_{iN} V_{cN}^2 V_{jN}}{2\lambda_N} (1 - e^{-2\lambda_N T_f}) \sim e^{-2\lambda_N T_f}$. As a result, the maximal eigenvalue of $\mathbf{H}$ grows exponentially with T_f: $\eta_{\max} \sim e^{-2\lambda_N T_f}$ so that $E_{\min} \sim e^{2\lambda_N T_f}$. Since $\lambda_N < 0$, the lower bound of the energy cost vanishes exponentially with the control time T_f. In the borderline case where $\mathbf{A}$ is semi-PD, i.e., $\lambda_\alpha > 0$ for $\alpha = 1, 2, \ldots, N-1$ and $\lambda_N = 0$, the behavior of $\mathbf{H}$ can be characterized as $H_{ij} \approx \lim_{\lambda_N \to 0} \frac{V_{iN} V_{cN}^2 V_{jN}}{2\lambda_N} (1 - e^{-2\lambda_N T_f}) \sim T_f^{-1}$.

Our theoretical estimates for the lower bound $E_{\min}$ of the energy cost can be summarized as

$$E_{\min} \begin{cases} \approx T_f^{-1} & \text{small } T_f \\ \approx \frac{1}{[(\mathbf{A}+\mathbf{A}^T)^{-1}]_{cc}} & \text{large } T_f, \ \mathbf{A} \text{ is PD} \\ \xrightarrow[\sim \exp(2\lambda_N T_f)]{\sim T_f^{-1}} 0 & \text{large } T_f, \ \mathbf{A} \text{ is } \frac{\text{semi PD}}{\text{not PD}} \end{cases}. \tag{7}$$

Numerical support for Eq. (7) is shown in Fig. 1. We use scale-free networks generated by the Barabási-Albert (BA) model [21] and Erdös-Réyni (ER) type of random networks [22]. The link weights are randomly generated from the uniform interval [0.5, 1.5]. The linear nodal dynamics are set as $a_{ii} = -(a + s_i)$, where

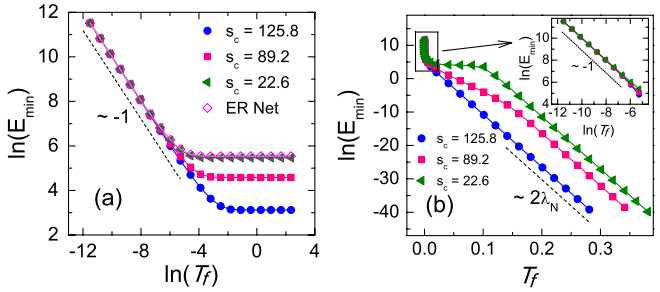

FIG. 1 (color online). Lower bound of the energy cost $E_{\min} \equiv 1/\eta_{\max}$ versus the control time T_f. All networks are weighted BA scale-free networks except one weighted ER random network in (a), with the same size $N = 500$ and $\langle s \rangle = 20$. The s_c denotes the strength of directly controlled node. In (a) $a = -150$ which makes $\mathbf{A}$ PD. In (b) $a = -50$ thus $\mathbf{A}$ is not PD. The dashed line in the semilog plot in (b) has a slope $2\lambda_N$. The symbols represent the same quantities calculated numerically and the solid lines represent the results from the estimation $\eta_{\max} \approx \mathrm{Tr}[\mathbf{H}]$.

$s_i = \sum_{j=1,j\neq i}^{N} a_{ij}$ is the strength of node i, and a is such a tunable parameter that one can conveniently change $\mathbf{A}$ between positive and negative definite. We note that other node-dependent settings of a_{ii} will not affect our results. Using the method proposed in [8] one can find the weighted network is controllable, except some pathological link-weights sets of measure zero, by any single driver node. We numerically compute the lower bound according to Eqs. (4) and (5). From Figs. 1(a) and the inset of 1(b), we see that, for the small T_f regime, $E_{\min}$ decays as a power law T_f^{-1}, regardless of $\mathbf{A}$ and c, agreeing with our theoretical result. In the large T_f regime, the behavior of $E_{\min}$ is determined by the signs of the eigenvalues of $\mathbf{A}$. In particular, if the eigenvalues are all positive, the dynamics in the absence of control, i.e., $\dot{\mathbf{x}}_t = \mathbf{A}\mathbf{x}_t$, will force the nodal states to depart away from the zero state. Thus, even given sufficiently large time, one has to consume some amount of energy to steer the nodes back. As shown in Fig. 1(a), $E_{\min}$ converges to a constant value as T_f is increased, which agrees with our predicted value $1/[(\mathbf{A} + \mathbf{A}^T)^{-1}]_{cc}$. In contrast, if $\mathbf{A}$ is not PD, $E_{\min}$ vanishes exponentially, as shown in Fig. 1(b). The corresponding exponent is $2\lambda_N$, which is consistent with our theoretical estimate in Eq. (7) as well.

We now turn to the upper bound of the energy cost $E_{\max} \equiv 1/\eta_{\min}$. As indicated by Eq. (6), most elements of the matrix $\mathbf{H}$ are small, especially for the small T_f regime. Consequently, $\mathbf{H}$ is generally ill-conditioned [20] and its minimal eigenvalue is typically very small (though positive). Thus, to control a large-size network, $E_{\max}$ can be very large. The underlying physical reason is that, when only one node is subject to control, the effect on other nodes will not be direct but instead will be indirect through various paths on the network. The end result is that we need to steer the whole system in the state space by following highly circuitous, though smooth, routes [17], a process that requires a large amount of energy.

Typical results computed from Eqs. (4) and (5) are shown in Figs. 2(a)–2(c). For small T_f, the upper bound $E_{\max}$ exhibits power-law decay, similar to the behavior of the lower bound, but the decay exponent for $E_{\max}$ assumes a much larger value that is independent of a and c [see Fig. 2(d)]. For large T_f, $E_{\max}$ will converge to a constant value if $\mathbf{A}$ is not negative definite (ND), or will vanish exponentially if $\mathbf{A}$ is ND. The corresponding exponent is given by $2\lambda_1$, where λ_1 is the least negative eigenvalue of $\mathbf{A}$, as shown in Fig. 2(e). This is due to the fact that, in the large T_f limit, the behavior of H_{ij}^{-1} is dominated by the mode with the least negative eigenvalue λ_1, which contributes the slowest increase to H_{ij}. As a result, we have $E_{\max} \sim [\mathbf{H}^{-1}]_{ij} \sim H_{ij}^{-1} \sim \frac{2\lambda_1}{(1-\exp(-2\lambda_1 T_f))} \sim e^{2\lambda_1 T_f}$. In the borderline case, i.e., $\mathbf{A}$ is semi ND, the upper bound decays according to T_f^{-1}: $E_{\max} \sim \lim_{\lambda_1 \to 0} \frac{2\lambda_1}{(1-\exp(-2\lambda_1 T_f))} \sim T_f^{-1}$. Such a behavior in both $E_{\max}$ and $E_{\min}$ has been numerically verified [17].

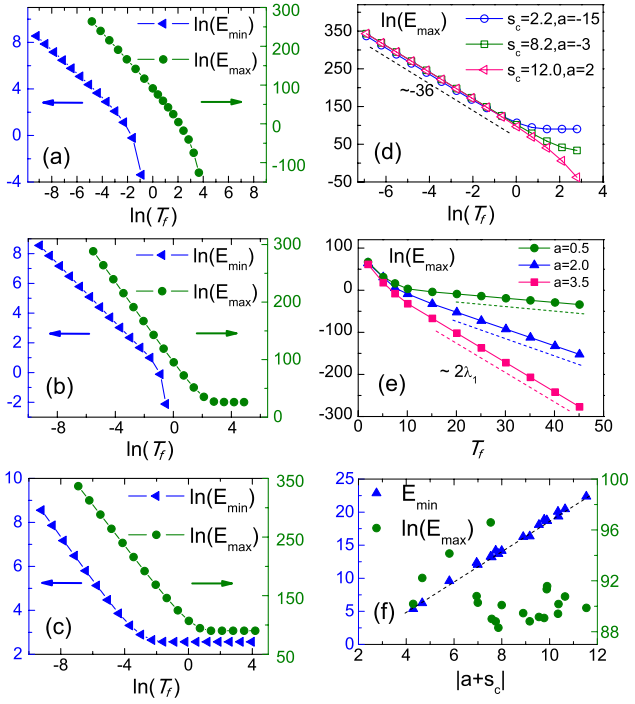

FIG. 2 (color online). Upper bound of the control energy cost $E_{\max} \equiv 1/\eta_{\min}$ for a weighted BA network with 20 nodes. In (a), $a = 2$ thus $\mathbf{A}$ is ND. In (b), $a = -5$. In (c), $a = -20$ so that $\mathbf{A}$ is PD. In (a)–(c), (●) represent the upper bound $E_{\max}$ while (◀) represent the corresponding lower bound $E_{\min}$ (included for comparison). In (d) the decaying behavior of $E_{\max}$ is shown for different s_c and a values. The dashed line has a slope -36. In (e) the exponential decay of $E_{\max}$ for large T_f is plotted for different values of a. The slopes of dashed lines are $2\lambda_1$, respectively. In (f) the constant values of the energy cost in (c) are shown as a function of $|a + s_c|$. The slope of the dashed line is 2.

The results for the upper bound can be summarized as

$$E_{\max}\begin{cases}\approx T_f^{-\theta}\ (\theta \gg 1) & \text{small } T_f \\ = \varepsilon(\mathbf{A}, c) & \text{large } T_f,\ \mathbf{A} \text{ is not ND} \\ \xrightarrow[\sim \exp(2\lambda_1 T_f)]{\sim T_f^{-1}} 0 & \text{large } T_f,\ \mathbf{A} \text{ is } \frac{\text{semi ND}}{\text{ND}}\end{cases}, \quad (8)$$

where $\varepsilon(\mathbf{A}, c)$ denotes a positive value that depends on the matrix $\mathbf{A}$ and the controlled node c. For the constant value of the lower bound as described in Eq. (7), one may approximate $1/[(\mathbf{A}+\mathbf{A}^T)^{-1}]_{cc} \approx 2a_{cc}$ so that $E_{\min}$ is proportional to $|a + s_c|$. However, as shown in Fig. 2(f), there appears no proportional relationship between the constant value $\varepsilon(\mathbf{A}, c)$ of $E_{\max}$ and a_{cc} of the controlled node. This indicates that directly controlling a node with larger degree does not generally result in less energy cost.

Actually, when the system matrix $\mathbf{A}$ is PD and the control time $T_f \to \infty$, Eq. (6) reduces to $H_{ij}^{\infty} = \sum_{\alpha=1}^{N}\sum_{\beta=1}^{N}\frac{V_{i\alpha}V_{c\alpha}V_{c\beta}V_{j\beta}}{\lambda_\alpha+\lambda_\beta}$ which is the solution of $\mathbf{A}\mathbf{H}^{\infty} + \mathbf{H}^{\infty}\mathbf{A}^T = \mathbf{B}\mathbf{B}^T$ and can be naturally interpreted as dynamical correlation [23], between nodes i and j with respect to controlled (driver) node c. So $\varepsilon(\mathbf{A}, c)$ is the inverse of the smallest eigenvalue of the correlation matrix $\mathbf{H}^{\infty}$. From this point of view, two indications come out immediately. First, to find optimal driver node in a network, one should consider the node viewing from which the rest nodes are most dissimilar. The reason is that, controlling a central hub node, though may transmit external signals fast, can induce starlike structure which makes the rest nodes more similar to each other. When nodes are more structurally similar, they tend to have more similar dynamical correlations with other nodes so that the corresponding rows in $\mathbf{H}^{\infty}$ become more similar. As a consequence, the smallest eigenvalue of $\mathbf{H}^{\infty}$ will be less. In other words, we have to consume more energy to independently steer similar nodes in order to fully control the network. Second, for randomized networks, the more heterogeneous the node degrees, the higher the energy cost of control (see Sec. III of [17]). Take randomized BA and ER networks, for example, we compare the values of $\varepsilon(\mathbf{A}, c)$, i.e., $\varepsilon_{\mathrm{BA}}$ and $\varepsilon_{\mathrm{ER}}$ in Fig. 3(a). It shows that the upper bound of energy cost for controlling BA networks is much larger than that for controlling ER networks.

We have also studied the energy cost associated with the control scheme proposed in a recent work [9], i.e., controlling more than one node by a common controller. Figure 3(b) shows the effect of n_c, the number of directly controlled nodes, on the energy cost, which reveals that controlling more nodes will induce smaller value of the lower energy bound. This, however, does not hold for the upper bound. In fact, adding a node with large degree into the directly controlled node set may drastically increase the energy cost. This result is consistent, to a certain degree, with that found in Ref. [8] which shows the driver nodes tend to avoid the high-degree nodes.

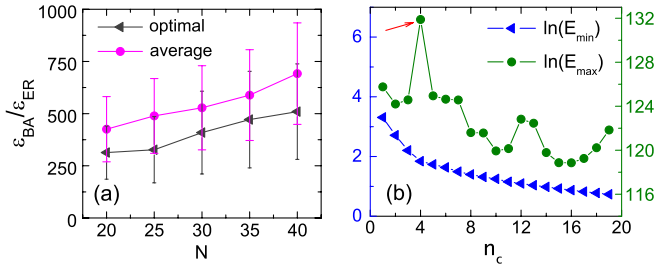

FIG. 3 (color online). (a) The ratio $\varepsilon_{\mathrm{BA}}/\varepsilon_{\mathrm{ER}}$ for different network size N. In order to eliminate the effects of nodal dynamics and strength, we fix the values of a_{ii} and $\langle s\rangle$. The results include the ratio for optimal driver node (◀) and the ratio of averaging over different driver nodes (●). The error bars are caused by different generations of network topology and link weights. (b) $E_{\min}$ (◀, left) and $E_{\max}$ (●, right) versus n_c, the number of directly controlled nodes. The dot pointed by the arrow corresponds to the node with largest degree in the network.

It is noteworthy that our results can be easily generalized to weighted directed networks. If a network is controllable by one driver node, the eigenvalues of the corresponding system matrix $\mathbf{A}$ are nondegenerate [16] though they may not all be real. Thus we have $\mathbf{A} = \mathbf{V}\mathbf{S}\mathbf{V}^{-1}$ where $\mathbf{S} = \mathrm{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_N\}$ with descending order of the real part $\mathrm{Re}\lambda_1 \geq \mathrm{Re}\lambda_2 \geq \ldots \geq \mathrm{Re}\lambda_N$. Similarly, $e^{\mathbf{A}t} = \mathbf{V}e^{\mathbf{S}t}\mathbf{V}^{-1}$ and $e^{\mathbf{A}^T t} = (\mathbf{V}^{-1})^T e^{\mathbf{S}t}\mathbf{V}^T$. As a consequence, Eq. (6) is replaced by $H_{ij} = \sum_{\alpha=1}^{N}\sum_{\beta=1}^{N}\frac{V_{i\alpha}(V^{-1})_{\alpha c}(V^{-1})_{\beta c}V_{j\beta}}{\lambda_\alpha+\lambda_\beta}\times(1 - e^{-(\lambda_\alpha+\lambda_\beta)T_f})$. Therefore, the scaling laws in Eqs. (7) and (8) keep unchanged while the decaying exponents are replaced by $2\,\mathrm{Re}\lambda_N$ and $2\,\mathrm{Re}\lambda_1$, respectively. Moreover, for large T_f and PD $\mathbf{A}$, the constant in Eq. (7) is still proportional to $2a_{cc}$ by using first-order approximation in [23].

In conclusion, we have reduced the complexity of the fundamental problem of control cost from the complicated and intractable Gramian matrix to the simple system matrix which is directly related to the network structure. Our results have revealed that energy cost of controlling complex networks has different scaling behaviors with control time in two time scales, separated by the characteristic time, $\frac{1}{2|\mathrm{Re}\lambda_N|}$ and $\frac{1}{2|\mathrm{Re}\lambda_1|}$ for the lower and the upper bound, respectively. In the small-time regime, setting a relatively longer time for control always leads to less energy cost. While, in the large-time regime, there exists the situation where we cannot reduce the energy cost even given much more time. Furthermore, our results indicate that the lower (upper) bound of energy cost is less when controlling a randomized network with heterogeneous (homogeneous) node degrees. These implications are important when considering the trade-off between the energy cost and the control time, which may find applications not only for classical [5,8] but also for biological [11,12,19] and quantum [24] networks. Although we have given some heuristics, a method to choose an optimal control node

set for minimizing the energy cost is lacking, which is a promising future work.

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